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THE COMPLETENESS OF THE FIRST-ORDER FUNCTIONAL CALCULUS

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Although several proofs have been published showing the completeness of the propositional calculus (cf. Quine (1)[2]), for the first-order functional calculus only the original completeness proof of Gödel (2) and a variant due to Hilbert and Bernays have appeared. Aside from novelty and the fact that it requires less formal development of the system from the axioms, the new method of proof which is the subject of this paper possesses two advantages. In the first place an important property of formal systems which is associated with completeness can now be generalized to systems containing a non-denumerable infinity of primitive symbols. While this is not of especial interest when formal systems are considered as *logics*—i.e., as means for analyzing the structure of languages—it leads to interesting applications in the field of abstract algebra. In the second place the proof suggests a new approach to the problem of completeness for functional calculi of higher order. Both of these matters will be taken up in future papers.

The system with which we shall deal here will contain as primitive symbols

$$(\quad) \quad \supset \quad f \quad ,$$

and certain sets of symbols as follows:

(i) *propositional symbols* (some of which may be classed as *variables*, others as *constants*), and among which the symbol "f" above is to be included as a constant;

(ii) for each number $n = 1, 2, \cdots$ a set of *functional symbols of degree n* (which again may be separated into *variables* and *constants*); and

(iii) *individual symbols* among which *variables* must be distinguished from *constants*. The set of variables must be infinite.

Elementary well-formed formulas are the propositional symbols and all formulas of the form $G(x_1, \cdots, x_n)$ where G is a functional symbol of degree n and each x_i is an individual symbol.

Well-formed formulas (wffs) consist of the elementary well-formed formulas together with all formulas built up from them by repeated application of the following methods:

(i) If A and B are wffs so is $(A \supset B)$;

(ii) If A is a wff and x an individual variable then $(x)A$ is a wff. Method (ii) for forming wffs is called *quantification with respect to the variable x*. Any occurrence of the variable x in the formula $(x)A$ is called *bound*. Any occurrence of a symbol which is not a bound occurrence of an individual variable according to this rule is called *free*.

In addition to formal manipulation of the formulas of this system we shall be concerned with their *meaning* according to the following interpretation. The propositional constants are to denote one of the truth values, T or F, the symbol "f" denoting F, and the propositional variables are to have the set of these truth values as their range. Let an arbitrary set, I, be specified as a domain of individuals, and let each individual constant denote a particular element of this domain while the individual variables have I as their range. The functional constants (variables) of degree n are to denote (range over) subsets of the set of all ordered n-tuples of I. $G(x_1, \cdots, x_n)$ is to have the value T or F according as the n-tuple $\langle x_1, \cdots, x_n \rangle$ of individuals is or is not in the set G; $(A \supset B)$ is to have the value F if A is T and B is F, otherwise T; and $(x)A$ is to have the value T just in case A has the value T for every element x in I.[3]

If A is a wff, I a domain, and if there is some assignment of denotations to the constants of A and of values of the appropriate kind to the variables with free occurrences in A, such that for this assignment A takes on the value T according to the above interpretation, we say that A is *satisfiable with respect to I*. If *every* such assignment yields the value T for A we say that A is *valid with respect to I*. A is *valid* if it is valid with respect to every domain. We shall give a set of axioms and formal rules of inference adequate to permit formal proofs of every valid formula.

Before giving the axioms, however, we describe certain rules of abbreviation which we use to simplify the appearance of wffs and formula schemata. If A is any wff and x any individual variable we write

$$\sim A \quad \text{for} \quad (A \supset f),$$

$$(\exists x)A \quad \text{for} \quad \sim(x)\sim A.$$

From the rules of interpretation it is seen that $\sim A$ has the value T or F according as A has the value F or T, while $(\exists x)A$ denotes T just in case there is some individual x in I for which A has the value T.

Furthermore we may omit outermost parentheses, replace a left parenthesis by a dot omitting its mate at the same time if its mate comes at the end of the formula (except possibly for other right parentheses), and put a sequence of wffs separated by occurrences of "$\supset$" when association to the left is intended. For example,

$$A \supset B \supset \, . \, C \supset D \supset E \quad \text{for} \quad ((A \supset B) \supset ((C \supset D) \supset E)),$$

where A, B, C, D, E may be wffs or abbreviations of wffs.

If A, B, C are any wffs, the following are called *axioms*:

1. $C \supset \, . \, B \supset C$
2. $A \supset B \supset \, . \, A \supset (B \supset C) \supset \, . \, A \supset C$
3. $A \supset f \supset f \supset A$
4. $(x)(A \supset B) \supset \, . \, A \supset (x)B$, where x is any individual variable with no free occurrence in A.

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[3] A more precise, syntactical account of these ideas can be formulated along the lines of Tarski (3). But this semantical version will suffice for our purposes.

5. $(x)A \supset B$, where x is any individual variable, y any individual symbol, and B is obtained by substituting y for each free occurrence of x in A, provided that no free occurrence of x in A is in a well-formed part of A of the form $(y)C$.

There are two formal rules of inference:

I (*Modus Ponens*). To infer B from any pair of formulas A, $A \supset B$.

II (*Generalization*). To infer $(x)A$ from A, where x is any individual variable.

A finite sequence of wffs is called a *formal proof from assumptions* Γ, where Γ is a set of wffs, if every formula of the sequence is either an axiom, an element of Γ, or else arises from one or two previous formulas of the sequence by *modus ponens* or generalization, except that no variable with a free occurrence in some formula of Γ may be generalized upon. If A is the last formula of such a sequence we write $\Gamma \vdash A$. Instead of $\{\Gamma, A\} \vdash B$ ($\{\Gamma, A\}$ denoting the set formed from Γ by adjoining the wff A), we shall write $\Gamma, A \vdash B$. If Γ is the empty set we call the sequence simply a *formal proof* and write $\vdash A$. In this case A is called a *formal theorem*. Our object is to show that every valid formula is a formal theorem, and hence that our system of axioms and rules is *complete*.

The following theorems about the first-order functional calculus are all either well-known and contained in standard works, or else very simply derivable from such results. We shall use them without proof here, referring the reader to Church (4) for a fuller account.

III (*The Deduction Theorem*). If $\Gamma, A \vdash B$ then $\Gamma \vdash A \supset B$ (for any wffs A, B and any set Γ of wffs).

6. $\vdash B \supset f \supset . B \supset C$
7. $\vdash B \supset . C \supset f \supset . B \supset C \supset f$
8. $\vdash (x)(A \supset f) \supset . (\exists x)A \supset f$
9. $\vdash (x)B \supset f \supset . (\exists x)(B \supset f)$.

IV. If Γ is a set of wffs no one of which contains a free occurrence of the individual symbol u, if A is a wff and B is obtained from it by replacing each free occurrence of u by the individual symbol x (none of these occurrences of x being bound in B), then if $\Gamma \vdash A$, also $\Gamma \vdash B$.

This completes our description of the formal system; or, more accurately, of a class of formal systems, a certain degree of arbitrariness having been left with respect to the nature and number of primitive symbols.

Let S_0 be a particular system determined by some definite choice of primitive symbols. A set Λ of wffs of S_0 will be called *inconsistent* if $\Lambda \vdash f$, otherwise *consistent*. A set Λ of wffs of S_0 will be said to be *simultaneously satisfiable* in some domain I of individuals if there is some assignment of denotations (values) of the appropriate type to the constants (variables) with free occurrences in formulas of Λ, for which each of these formulas has the value T under the interpretation previously described.

THEOREM.   *If Λ is a set of formulas of S_0 in which no member has any occurrence of a free individual variable, and if Λ is consistent, then Λ is simultaneously satisfiable in a domain of individuals having the same cardinal number as the set of primitive symbols of S_0.*

We shall carry out the proof for the case where S_0 has only a denumerable infinity of symbols, and indicate afterward the simple modifications needed in the general case.

Let u_{ij} $(i, j = 1, 2, 3, \cdots)$ be symbols not occurring among the symbols of S_0.   For each i $(i = 1, 2, 3, \cdots)$ let S_i be the first-order functional calculus whose primitive symbols are obtained from those of S_{i-1} by adding the symbols u_{ij} $(j = 1, 2, 3, \cdots)$ as individual constants.   Let S_ω be the system whose symbols are those appearing in any one of the systems S_i.   It is easy to see that the wffs of S_ω are denumerable, and we shall suppose that some particular enumeration is fixed on so that we may speak of the first, second, $\cdots$, nth, $\cdots$ formula of S_ω in the standard ordering.

We can use this ordering to construct in S_0 a maximal consistent set of cwffs, Γ_0, which contains the given set Λ.   (We use "cwff" to mean *closed* wff: a wff *which contains no free occurrence of any individual variable*.) Γ_0 is maximal consistent in the sense that if A is any cwff of S_0 which is not in Γ_0, then $\Gamma_0, A \vdash f$; but not $\Gamma_0 \vdash f$.

To construct Γ_0 let Γ_{00} be Λ and let B_1 be the first (in the standard ordering) cwff A of S_0 such that $\{\Gamma_{00}, A\}$ is consistent.   Form Γ_{01} by adding B_1 to Γ_{00}.   Continue this process as follows.   Assuming that Γ_{0i} and B_i have been found, let B_{i+1} be the first cwff A (of S_0) after B_i, such that $\{\Gamma_{0i}, A\}$ is consistent; then form Γ_{0i+1} by adding B_{i+1} to Γ_{0i}.   Finally let Γ_0 be composed of those formulas appearing in any Γ_{0i} $(i = 0, 1, \cdots)$.   Clearly Γ_0 *contains* Λ. Γ_0 is *consistent*, for if $\Gamma_0 \vdash f$ then the formal proof of f from assumptions Γ_0 would be a formal proof of f from some finite subset of Γ_0 as assumptions, and hence for some $i(i = 0, 1, \cdots)$ $\Gamma_{0i} \vdash f$ contrary to construction of the sets of Γ_{0i}.   Finally, Γ_0 is *maximal* consistent because if A is a cwff of S_0 such that $\{\Gamma_0, A\}$ is consistent then surely $\{\Gamma_{0i}, A\}$ is consistent for each i; hence A will appear in some Γ_{0i} and so in Γ_0.

Having obtained Γ_0 we proceed to the system S_1 and form a set Γ_1 of its cwffs as follows.   Select the first (in the standard ordering) cwff of Γ_0 which has the form $(\exists x)A$ (unabbreviated: $((x)(A \supset f) \supset f)$), and let A' be the result of substituting the symbol u_{11} of S_1 for all free occurrences of the variable x in the wff A.   The set $\{\Gamma_0, A'\}$ must be a consistent set of cwffs of S_1.   For suppose that $\Gamma_0, A' \vdash f$.   Then by III (the Deduction Theorem), $\Gamma_0 \vdash A' \supset f$; hence by IV, $\Gamma_0 \vdash A \supset f$; by II, $\Gamma_0 \vdash (x)(A \supset f)$; and so by 8 and I, $\Gamma_0 \vdash (\exists x)A \supset f$.   But by assumption $\Gamma_0 \vdash (\exists x)A$.   Hence modus ponens gives $\Gamma_0 \vdash f$ contrary to the construction of Γ_0 as a consistent set.

We proceed in turn to each cwff of Γ_0 having the form $(\exists x)A$, and for the jth of these we add to Γ_0 the cwff A' of S_1 obtained by substituting the constant u_{1j} for each free occurrence of the variable x in the wff A.   Each of these adjunctions leaves us with a consistent set of cwffs of S_1 by the argument above.

Finally, after all such formulas A' have been added, we enlarge the resulting set of formulas to a maximal consistent set of cwffs of S_1 in the same way that Γ_0 was obtained from Λ in S_0. This set of cwffs we call Γ_1.

After the set Γ_i has been formed in the system S_i we construct Γ_{i+1} in S_{i+1} by the same method used in getting Γ_i from Γ_0 but using the constants u_{i+1j} ($j = 1, 2, 3, \cdots$) in place of u_{1j}. Finally we let Γ_ω be the set of cwffs of S_ω consisting of all those formulas which are in any Γ_1. It is easy to see that Γ_ω possesses the following properties:

i) Γ_ω is a maximal consistent set of cwffs of S_ω.

ii) If a formula of the form $(\exists x)A$ is in Γ_ω then Γ_ω also contains a formula A' obtained from the wff A by substituting some constant u_{ij} for each free occurrence of the variable x.

Our entire construction has been for the purpose of obtaining a set of formulas with these two properties; they are the only properties we shall use now in showing that the elements of Γ_ω are simultaneously satisfiable in a denumerable domain of individuals.

In fact we take as our domain I simply the set of individual constants of S_ω, and we assign to each such constant (considered as a symbol in an interpreted system) itself (considered as an individual) as denotation. It remains to assign values in the form of truth-values to propositional symbols, and sets of ordered n-tuples of individuals to functional symbols of degree n, in such a way as to lead to a value T for each cwff of Γ_ω.

Every propositional symbol, A, of S_0 is a cwff of S_ω; we assign to it the value T or F according as $\Gamma_\omega \vdash A$ or not. Let G be any functional symbol of degree n. We assign to it the class of those n-tuples $\langle a_1, \cdots, a_n \rangle$ of individual constants such that $\Gamma_\omega \vdash G(a_1, \cdots, a_n)$.

This assignment determines a unique truth-value for each cwff of S_ω under the fundamental interpretation prescribed for quantification and "$\supset$". (We may note that the symbol "f" is assigned F in agreement with that interpretation since Γ_ω is consistent.) We now go on to show the

LEMMA: *For each cwff A of S_ω the associated value is* T *or* F *according as* $\Gamma_\omega \vdash A$ *or not.*

The proof is by induction on the length of A. We may notice, first, that if we do not have $\Gamma_\omega \vdash A$ for some cwff A of S_ω then we do have $\Gamma_\omega \vdash A \supset f$. For by property i) of Γ_ω we would have $\Gamma_\omega, A \vdash f$ and so $\Gamma_\omega \vdash A \supset f$ by III.

In case A is an elementary cwff the lemma is clearly true from the nature of the assignment.

Suppose A is $B \supset C$. If C has the value T, by induction hypothesis $\Gamma_\omega \vdash C$; then $\Gamma_\omega \vdash B \supset C$ by 1 and I. This agrees with the lemma since $B \supset C$ has the value T in this case. Similarly, if B has the value F we do not have $\Gamma_\omega \vdash B$ by induction hypothesis. Hence $\Gamma_\omega \vdash B \supset f$, and $\Gamma_\omega \vdash B \supset C$ by 6 and I. Again we have agreement with the lemma since $B \supset C$ has the value T in this case also. Finally if B and C have the values T and F respectively, so that (induction hypothesis) $\Gamma_\omega \vdash B$ while $\Gamma_\omega \vdash C \supset f$, we must show that $\Gamma_\omega \vdash B$

$\supset C$ does not hold (since $B \supset C$ has the value F in this case). But by 7 and two applications of I we conclude that $\Gamma_\omega \vdash B \supset C \supset f$. Now we see that if $\Gamma_\omega \vdash B \supset C$ then $\Gamma_\omega \vdash f$ by I, contrary to the fact that Γ_ω is consistent (property i).

Suppose A is $(x)B$. If $\Gamma_\omega \vdash (x)B$ then (by 5 and I) $\Gamma_\omega \vdash B'$, where B' is obtained by replacing all free occurrences of x in B by some (arbitrary) individual constant. That is, (induction hypothesis) B has the value T for every individual x of I; therefore A has the value T and the lemma is established in this case. If, on the other hand, we do not have $\Gamma_\omega \vdash (x)B$, then $\Gamma_\omega \vdash (x)B \supset f$ whence (by 9, I) $\Gamma_\omega \vdash (\exists x)(B \supset f)$. Hence, by property ii of Γ_ω, for some individual constant u_{ij} we have $\Gamma_\omega \vdash B' \supset f$, where B' is obtained from B by replacing each free occurrence of x by u_{ij}. Hence for this u_{ij} we cannot have $\Gamma_\omega \vdash B'$ else $\Gamma_\omega \vdash f$ by I contrary to the fact that Γ_ω is consistent (property i). That is, by induction hypothesis, B has the value F for at least the one individual u_{ij} of I and so $(x)B$ has the value F as asserted by the lemma for this case.

This concludes the inductive proof of the lemma. In particular the formulas of Γ_ω all have the value T for our assignment and so are simultaneously satisfiable in the denumerable domain I. Since the formulas of Λ are included among those of Γ_ω our theorem is proved for the case of a system S_o whose primitive symbols are denumerable.

To modify the proof in the case of an arbitrary system S_o it is only necessary to replace the set of symbols u_{ij} by symbols $u_{i\alpha}$, where i ranges over the positive integers as before but α ranges over a set with the same cardinal number as the set of primitive symbols of S_0; and to fix on some particular well-ordering of the formulas of the new S_ω in place of the standard enumeration employed above. (Of course the axiom of choice must be used in this connection.)

The completeness of the system S_o is an immediate consequence of our theorem.

COROLLARY 1: *If A is a valid* wff *of S_0 then $\vdash A$.*
First consider the case where A is a cwff. Since A is valid $A \supset f$ has the value F for any assignment with respect to any domain; i.e., $A \supset f$ is not satisfiable. By our theorem it is therefore inconsistent: $A \supset f \vdash f$. Hence $\vdash A \supset f \supset f$ by III and $\vdash A$ by 3 and I.

The case of wff A' which contains some free occurrence of an individual variable may be reduced to the case of the cwff A (the *closure* of A') obtained by prefixing to A' universal quantifiers with respect to each individual variable with free occurrences in A' (in the order in which they appear). For it is clear from the definition of validity that if A' is valid so is A. But then $\vdash A$. From which we may infer $\vdash A'$ by successive applications of 5 and I.

COROLLARY 2: *Let S_0 be a functional calculus of first order and* **m** *the cardinal number of the set of its primitive symbols. If Λ is a set of* cwffs *which is simultaneously satisfiable then in particular Λ is simultaneously satisfiable in some domain of cardinal* **m**.

This is an immediate consequence of our theorem and the fact that if Λ is simultaneously satisfiable it must also be consistent (since rules of inference

preserve the property of having the value T for any particular assignment in any domain, and so could not lead to the formula f). For the special case where $\mathbf{m}$ is $\aleph_0$ this corollary is the well-known Skolem-Löwenheim result (5). It should be noticed, for this case, that the assertion of a set of cwffs Λ can no more compel a domain to be finite than non-denumerably infinite: there is always a denumerably infinite domain available. There are also always domains of any cardinality greater than $\aleph_0$ in which a consistent set Λ is simultaneously satisfiable, and sometimes finite domains. However for certain Λ no finite domain will do.

Along with the truth functions of propositional calculus and quantification with respect to individual variables the first-order functional calculus is sometimes formulated so as to include the notion of equality as between individuals. Formally this may be accomplished by singling out some functional constant of degree 2, say Q, abbreviating $Q(x, y)$ as $x = y$ (for individual symbols x, y), and adding the axiom schemata

E1. $x = x$

E2. $x = y \supset . A \supset B$, where B is obtained from A by replacing some free occurrence of x by a free occurrence of y.

For a system S_0' of this kind our theorem holds if we replace "the same cardinal number as" by "a cardinal number not greater than," where the definition of "simultaneously satisfiable" must be supplemented by the provision that the symbol "$=$" shall denote the relation of equality between individuals. To prove this we notice that a set of cwffs Λ in the system S_0' may be regarded as a set of cwffs (Λ, E_1, E_2) in the system S_0, where E_1 is the set of closures of axioms E_i $(i = 1, 2)$. Since $E_1, E_2 \vdash x = y \supset y = x$ and $E_1, E_2 \vdash x = y \supset . y = z \supset x = z$ we see that the assignment which gives a value T to each formula of Λ, E_1, E_2 must assign some equivalence relation to the functional symbol Q. If we take the domain I' of equivalence classes determined by this relation over the original domain I of constants, and assign to each individual constant (as denotation) the class determined by itself, we are led to a new assignment which is easily seen to satisfy Λ (simultaneously) in S_0'.

A set of wffs may be thought of as a set of axioms determining certain domains as models; namely, domains in which the wffs are simultaneously satisfiable. For a first-order calculus containing the notion of equality we can find axiom sets which restrict models to be finite, unlike the situation for calculi without equality. More specifically, given any finite set of finite numbers there exist axiom sets whose models are precisely those domains in which the number of individuals is one of the elements of the given set. (For example, if the set of numbers is the pair $(1, 3)$ the single axiom

$$(x)(y)(x = y) \lor . (\exists x)(\exists y)(\exists z) . \sim (x = y) \land \sim (x = z)$$

$$\land \sim (y = z) \land (t) . t = x \lor t = y \lor t = z$$

will suffice, where $A \land B$, $A \lor B$ abbreviate $\sim(A \supset \sim B)$, $A \supset B \supset B$ respectively.) However, an axiom set which has models of arbitrarily large finite

cardinality must also possess an infinite model as one sees by considering the formulas

$$C_i: \quad (\exists x_1)(\exists x_2) \cdots (\exists x_i) \ . \sim(x_1 = x_2) \wedge \sim(x_1 = x_2) \cdots \wedge \sim(x_{i-1} = x_i).$$

Since by hypothesis any finite number of the C_i are simultaneously satisfiable they are consistent. Hence all the C_i are consistent and so simultaneously satisfiable—which can happen only in an infinite domain of individuals.

There are axiom sets with no finite models—namely, the set of all formulas C_i defined above. Every axiom set with an infinite model has models with arbitrary infinite cardinality. For if α, β range over any set whatever the set of all formulas $\sim(x_\alpha = x_\beta)$ for distinct α, β will be consistent (since the assumption of an infinite model guarantees consistency for any finite set of these formulas) and so can be simultaneously satisfied.

In simplified form the proof of our theorem and corollary 1 may be carried out for the propositional calculus. For this system the symbols u_{ij} and the construction of S_ω may be omitted, an assignment of values being made directly from Γ_0. While such a proof of the completeness of the propositional calculus is short compared with other proofs in the literature the latter are to be preferred since they furnish a constructive method for finding a formal proof of any given tautology, rather than merely demonstrate its existence.

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